Sign Patterns of Inverse Nonnegative Matrices*

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ABSTRACT

We determine here the $+, -, 0$ sign patterns which occur among the inverses of nonsingular, entrywise nonnegative matrices. These results complete a sequence of work which began with the characterization of all possible, $+, -$ sign patterns which occur among inverses of positive matrices, and the characterization in this general case is more involved to state than the earlier ones.

INTRODUCTION

In [3] the $+, -$ sign patterns which occur among the inverses of componentwise positive matrices were characterized. In [2], with the aid of [1], this was extended to identify those $+, -, 0$ patterns which occur among inverse positive matrices. The $+, -, 0$ result was similar to the $+, -$ one, in that restrictions involving complementary positive and negative blocks were replaced by restrictions involving complementary nonnegative and nonpositive blocks. This extension had also been noted independently (unpublished) by the authors of [3], but [2] also includes several nice additional equivalent conditions. (The paper [2] makes use of the observation that if a sign pattern admits a positive inverse, it also admits a doubly stochastic inverse.) The purpose of the current note is to complete the sequence and identify those $+, -, 0$ sign patterns which occur among nonsingular matrices whose inverses are componentwise nonnegative.

The inverse nonnegative case is both interestingly similar to, and different from, the inverse positive case. Among fully indecomposable patterns, it turns out that the inverse nonnegative patterns are the same as the inverse positive

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patterns (Theorem 1), but, in general, the inverse nonnegative patterns are notably different. For example, in dimension at least 2, the inverse positive patterns are (surprisingly) closed under negation (of all signs), but the inverse nonnegative patterns are not in general; and there is an asymmetry between $+$ and $-$ in the characterizing conditions (Theorem 2).

The characterization of inverse nonnegative sign patterns may readily be applied to pleasantly show, for example, that among irreducible matrices whose directed graphs have simple circuits of length no more than 2, the only monotone, positive-stable matrices are M-matrices [4].

BACKGROUND

An n-by-n matrix A is said to be *reducible* if A is similar via a permutation matrix to a matrix of the form

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \tag{1}$$

where the A_{ii}, $i=1,2$, are square and nonempty; otherwise, A is called *irreducible*. Two n-by-n matrices A and C are said to be *permutation equivalent* if there are permutation matrices P and Q such that

$$C = PAQ.$$

If the n-by-n matrix A is permutation equivalent to a matrix of the form (1), A is called *partly decomposable*; otherwise, A is called *fully indecomposable*. Note that A is fully indecomposable if and only if PA is irreducible for every permutation matrix P. If A is partly decomposable, then it is clear that A is permutation equivalent to a matrix of the form

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ 0 & A_{22} & \cdots & A_{2k} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & A_{kk} \end{bmatrix} \tag{2}$$

in which each A_{ii} is square and either fully indecomposable or a 1-by-1 0 matrix, $i=1,\ldots,k$. Neither the permutations P and Q nor the form (2) is necessarily unique, but that will not be a consideration herein.

We are interested in the pattern of signs ($+$, $-$, and 0) of a real matrix and, in particular, in all those patterns which occur among matrices which have componentwise nonnegative inverses. For this reason, we shall speak in terms of *sign-pattern matrices*, that is, matrices whose entries are $+$, $-$, or 0. Such a matrix $B=(b_{ij})$ naturally defines a class of real matrices, namely all those real matrices $A=(a_{ij})$ such that

$$a_{ij}>0 \quad\Leftrightarrow\quad b_{ij}=+,$$

$$a_{ij}<0 \quad\Leftrightarrow\quad b_{ij}=-,$$

$$a_{ij}=0 \quad\Leftrightarrow\quad b_{ij}=0.$$

We say that a sign pattern (matrix) B is *inverse nonnegative* (*inverse positive*), if there is a real matrix with an entrywise nonnegative (positive) inverse in the class defined by B. Our goal is to characterize the inverse nonnegative sign patterns, or, equivalently, those real matrices which, by virtue of the pattern of signs of their entries alone, have a chance of having a nonnegative inverse.

We note that all the concepts mentioned in the first paragraph of this section apply equally well to a sign-pattern matrix B and uniformly to the entire class of real matrices associated with B. Inequality signs also apply in an obvious way to a sign-pattern matrix B, so that $B>\ (\geqslant)\ 0$ means all entries of B are $+$ ($+$ or 0). For a real matrix $A=(a_{ij})$ inequality signs of course have the analogous meaning, so that $A>\ (\geqslant)\ 0$ means $a_{ij}>\ (\geqslant)\ 0$ for all i,j. The meaning of negation of a sign pattern $(-B)$ is also unambiguous ($+\to-$, $-\to+$, $0\to0$).

Often, our partitions of an n-by-n (either sign-pattern or real) matrix C will be nonstandard:

$$C=\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}, \tag{3}$$

where C_{11} is n_1-by-m_1, C_{12} is n_1-by-m_2, C_{21} is n_2-by-m_1, and C_{22} is n_2-by-m_2, with $n_1+n_2=n=m_1+m_2$ and n_1 not necessarily equal to m_1. Here we assume that $n_1m_1\neq0$ or $n_2m_2\neq0$ (i.e., at least one C_{ii} is not empty), but not necessarily both. More generally, suppose that $\alpha,\beta\subset\{1,2,\dots,n\}$ are index sets, and let $C(\alpha,\beta)$ denote that submatrix of the n-by-n matrix C which lies in the rows indicated by α and the columns indicated by β. The pair α,β of index sets is called a *proper* pair if at most one of α, β, α', and β' is empty. For any proper pair of index sets α,β, the matrix C is permutation equivalent to one which may be partitioned in the form (3) with

$$\begin{matrix} C_{11} = C(\alpha, \beta) & C_{12} = C(\alpha, \beta') \\ C_{21} = C(\alpha', \beta) & C_{22} = C(\alpha', \beta') \end{matrix} \tag{3'}$$

An n-by-n sign-pattern matrix B is called *complementary* (*strictly complementary*) if B is (permutation equivalent to) a matrix of the form (3′) with

$$B(\alpha, \beta') \leqslant (<) \, 0 \quad \text{and} \quad B(\alpha', \beta) \geqslant (>) \, 0.$$

If a pattern B has a row or column without a $-$ or a row or column without a $+$, note that it is automatically complementary; this is precisely the case in which exactly one of $\alpha, \beta, \alpha', \beta'$ is empty. Note further, however, that for $n = 1$, the concept of a complementary pattern does not apply, since it is not possible for a pair of index sets to be proper.

Since a sign-pattern matrix B is inverse nonnegative if and only if each sign pattern which is permutation equivalent to B is also, permutation equivalence is a natural tool and a natural setting in which to express a characterization. It is, by the way, immediate from the Frobenius-Konig theorem, for example, that all matrices in the class defined by the n-by-n sign-pattern matrix B are singular if and only if B is permutation equivalent to a matrix of the form (3) with $B_{21} = 0$ and $n_2 + m_1 > n$.

RESULTS AND PROOFS

For $n = 1$, it is clear that there is exactly one inverse nonnegative sign pattern, namely the pattern

$$(+).$$

Since the case $n = 1$ is atypical and unwieldy to include in general statements of results, we suppose, throughout the remainder of this paper, that $n \geqslant 2$.

An observation which tightens a portion of the necessity proof in [3] is the following

LEMMA. *Let B be an n-by-n inverse nonnegative sign pattern. If B is complementary, with*

$$B(\alpha, \beta') \leqslant 0 \quad \text{and} \quad B(\alpha', \beta) \geqslant 0$$

for a proper pair of index sets α, β, then for each $A \geqslant 0$, with A^{-1} in the class *defined by B, we have*

$$A(\beta', \alpha) = 0.$$

Proof. We may assume, without loss of generality, that B is partitioned in the form (3) satisfying (3′) and that A is correspondingly partitioned

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

with A_{ii} m_i-by-n_i, $i = 1,2$; i.e., A_{21} corresponds to $A(\beta', \alpha)$, etc. We know that A^{-1}, partitioned in the form (3), satisfies $A^{-1}{}_{12} \leqslant 0$ and $A^{-1}{}_{21} \geqslant 0$. From $A^{-1}A = I$, we have

$$A^{-1}{}_{11}A_{11} = I - A^{-1}{}_{12}A_{21},$$

$$-A^{-1}{}_{22}A_{21} = A^{-1}{}_{21}A_{11}.$$

Calculation then produces

$$A^{-1}\begin{bmatrix} A_{11} \\ -A_{21} \end{bmatrix} = \begin{bmatrix} I - 2A^{-1}{}_{12}A_{21} \\ 2A^{-1}{}_{21}A_{11} \end{bmatrix} \geqslant 0,$$

using which we may write

$$\begin{bmatrix} A_{11} \\ -A_{21} \end{bmatrix} = AA^{-1}\begin{bmatrix} A_{11} \\ -A_{21} \end{bmatrix} = A\begin{bmatrix} I - 2A^{-1}{}_{12}A_{21} \\ 2A^{-1}{}_{21}A_{11} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} + 2\left(A_{12}A^{-1}{}_{21}A_{11} - A_{11}A^{-1}{}_{12}A_{21}\right) \\ A_{21} + 2\left(A_{22}A^{-1}{}_{21}A_{11} - A_{21}A^{-1}{}_{12}A_{21}\right) \end{bmatrix} \geqslant 0,$$

because both factors in the third expression are nonnegative. Since $A_{21} \geqslant 0$ and $-A_{21} = A_{21} + 2(A_{22}A^{-1}{}_{21}A_{11} - A_{21}A^{-1}{}_{12}A_{21}) \geqslant 0$, we conclude $A_{21} = 0$. ∎

The fully indecomposable portion of the characterization of inverse nonnegative sign patterns is included in

THEOREM 1. *Suppose that B is an n-by-n fully indecomposable sign-pattern matrix. Then, B is inverse nonnegative if and only if B is not complementary.*

Proof. The stated condition is sufficient for inverse positivity (as shown in [2], where condition (ii) implies condition (i) of the theorem, for example) and therefore sufficient for inverse nonnegativity.

The demonstration of necessity, which completes the proof, is as follows. We show that if B is inverse nonnegative, and *can* be partitioned in the form (3) with $B_{12} \leqslant 0$ and $B_{21} \geqslant 0$, then B must be partly decomposable. Continuing the line of argument which demonstrated the lemma, application of the result of the lemma to the partitioned product $A^{-1}A$ yields

$$A^{-1}{}_{21}A_{11} = 0. \tag{4}$$

Since each of $A^{-1}{}_{21}$ and A_{11} is nonnegative, $A^{-1}{}_{21}$ is n_2-by-m_1, and A_{11} is m_1-by-n_1, (4) implies that at least a total of m_1 columns of $A^{-1}{}_{21}$ and rows of A_{11} are 0. Let p be the number of 0 rows of A_{11}, and q the number of 0 columns of $A^{-1}{}_{21}$, so that $p+q \geqslant m_1$. Since $A_{21}=0$ because of the lemma, A then has an (m_2+p)-by-n_1 submatrix which is 0, and A^{-1} has an n_2-by-q submatrix which is 0. Two well-known and simple observations which apply here are: (i) an n-by-n matrix is partly decomposable if and only if it has a 0 submatrix with a total of n rows and columns; and (ii) an n-by-n nonsingular matrix A is fully indecomposable if and only if A^{-1} is. If B were to be fully indecomposable, then A^{-1} and A would be, and then we would have to have

$$n_1 + m_2 + p < n \quad \text{and} \quad n_2 + q < n.$$

But this is impossible, since $n_1 + m_2 + p + n_2 + q \geqslant n_1 + n_2 + m_1 + m_2 = 2n$, and so B must be partly decomposable, as was to be shown. This completes the proof. ∎

It is further clear from the above proof that $p+q=m_1$ and that $n_1+m_1+p=n$ and $n_2+q=n$ in the event $B_{12} \leqslant 0$ and $B_{21} \geqslant 0$. Thus, if B is a complementary inverse nonnegative sign pattern, B must be partly decomposable. In fact, this can happen; for example,

$$\begin{pmatrix} + & - \\ 0 & + \end{pmatrix}$$

is an inverse nonnegative sign pattern. However, the only way that an inverse nonnegative pattern can be complementary is if the nonnegative block is actually 0. This means that, in the partly decomposable case, an asymmetry between + and − arises. Note that, in the fully indecomposable case, B is inverse nonnegative if and only if $-B$ is, but this no longer holds for partly decomposable patterns. (These observations will become more clear from the next theorem.)

It is worth emphasizing several observations, implicit so far, in

COROLLARY 1. *If the n-by-n sign pattern B is fully indecomposable, then the following are equivalent:*

(a) *B is inverse nonnegative;*
(b) *B is inverse positive;*
(c) *$-B$ is inverse nonnegative; and*
(d) *$-B$ is inverse positive.*

In the general case of a partly decomposable sign-pattern matrix, we may assume, without loss of generality, that the pattern is in the form (2). In this event, in order for the sign pattern B to be inverse nonnegative, each subpattern of the form

$$\begin{bmatrix} B_{ii} & B_{i,i+1} & \cdots & B_{ij} \\ 0 & B_{i+1,i+1} & \cdots & B_{i+1,j} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & B_{jj} \end{bmatrix}, \tag{5}$$

$1 \leqslant i \leqslant j \leqslant k$, must also be inverse nonnegative. (Recall that each B_{ii} is fully indecomposable.) The result covering the partly decomposable case is

THEOREM 2. *Suppose that B is an n-by-n partly decomposable sign-pattern matrix partitioned in the form* (2). *Then B is inverse nonnegative if and only if*

(a) *each (fully indecomposable) sign pattern B_{ii} is inverse nonnegative (i.e. noncomplementary), $i = 1,\dots,k$, and*
(b) *no submatrix of the form*

$$[B_{i,i+1},\dots,B_{ij}]$$

or

$$\begin{bmatrix} B_{ij} \\ \vdots \\ B_{j-1,j} \end{bmatrix}$$

is nonnegative and nonzero, $1 \leqslant i < j \leqslant j \leqslant k$.

Proof. We first demonstrate the necessity of conditions (a) and (b). The necessity of (a) follows from an immediate calculation, and that of (b) again employs the lemma as follows. If the pattern B is inverse nonnegative, then so are the subpatterns identified in (5). Suppose that

$$A = \begin{bmatrix} A_{ii} & \cdot & \cdot & \cdot & \cdot & A_{ij} \\ 0 & \cdot & & & & \cdot \\ \vdots & & \cdot & & & \vdots \\ \vdots & & & \cdot & & \vdots \\ \cdot & & & & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 & A_{jj} \end{bmatrix} \geqslant 0$$

is a real matrix with A^{-1} in the sign-pattern class of the subpattern of B of the form (5), and suppose that

$$\left[B_{i,i+1},\dots,B_{ij}\right] \geqslant 0,$$

for example. Application of the lemma, with the above matrix playing the role of B_{21} (after transposition), implies that

$$\left[A_{i,i+1},\dots,A_{ij}\right] = 0,$$

which, in turn, means that

$$\left[B_{i,i+1},\dots,B_{ij}\right] = 0.$$

The case of

$$\begin{bmatrix} B_{ij} \\ \vdots \\ B_{j-1,j} \end{bmatrix} \geqslant 0$$

is analogous, which completes the proof of necessity.

The demonstration of the sufficiency of conditions (a) and (b) utilizes the fact that each B_{ii} is inverse positive, since it is inverse nonnegative and fully indecomposable. The construction which verifies sufficiency is inductive on the number k in the form (2) for the pattern B. Let A be the real matrix in the class determined by B, and with nonnegative inverse, which we hope to construct; suppose A is partitioned as in (2), and let A^{-1}, with blocks $A^{-1}{}_{ij}$, be partitioned similarly. A key observation is that as long as (b) [as well as (a)] holds, and if neither

$$\left[B_{i,i+1},\ldots,B_{ij}\right]$$

nor

$$\begin{bmatrix} B_{ij} \\ \vdots \\ B_{j-1,j} \end{bmatrix}$$

is 0, then it can be arranged not only for $A^{-1}{}_{ij}$ to be nonnegative, but for it to be positive and arbitrarily large. If one of the above matrices is 0, then $A^{-1}{}_{ij}=0$. In case $k=2$, partitioned calculation yields

$$A^{-1}{}_{12} = -A^{-1}{}_{11}A_{12}A^{-1}{}_{22}. \tag{6}$$

Per assumption (b), either $A_{12}=0$ or it has at least one negative entry. Since that negative entry can be chosen arbitrarily large in absolute value and since $A^{-1}{}_{11}$ and $A^{-1}{}_{22}$ can be taken to be positive, it follows from (6) that either $A^{-1}{}_{12}=0$ (if $A_{12}=0$) or $A^{-1}{}_{12}$ can be taken to be positive and arbitrarily large. For arbitrary k, we have analogous to (6)

$$A^{-1}{}_{ik} = -\sum_{t=i}^{k-1} A^{-1}{}_{it}A_{tk}A^{-1}{}_{kk}, \qquad i=1,\ldots,k-1.$$

Again, either $A^{-1}{}_{ik}=0$ (because either $A_{ik}=0,\ldots,$ $A_{k-1,k}=0$; or $A_{i,i+1}=0,\ldots,$ $A_{ik}=0$, so that $A^{-1}{}_{i,i+1}=0,\ldots,$ $A^{-1}{}_{i,k-1}=0$, and $A_{i,k}=0$) or, assuming (b), we may inductively achieve $A^{-1}{}_{ik}>0$ and large. In any event, conditions (a) and (b) insure that B is inverse nonnegative, which completes the proof of sufficiency and of the theorem. ■

OBSERVATIONS

NOTE 1. Analogous to the equivalence of inverse nonnegativity and inverse positivity in the fully indecomposable case, we note that, for an inverse nonnegative sign pattern B, according to the construction used in the proof of Theorem 2, each block [corresponding to a partition of the form (2) of B] of a nonnegative matrix A, with A^{-1} in the class determined by B, must either be necessarily 0 or can be taken to be componentwise positive.

NOTE 2. Recognition of inverse nonnegative patterns can be carried out efficiently [$O(n^2)$]. First of all, full indecomposability can be recognized, or the form (2) achieved, efficiently by known, $O(n^2)$ means, and, if necessary, condition (b) of Theorem 2 can be verified straightforwardly. Recognition then revolves about the fully indecomposable case, for which, by Theorem 1, it suffices to check for complementarity. A procedure for this was suggested, for example, in [2] [condition (iii) of the theorem there], which turns out to be $O(n^2)$, although it requires handling a $2n$-by-$2n$ matrix.

NOTE 3. Strict complementarity can be checked in an especially simple way. Reorder the rows of a sign-pattern matrix B so that the number of $-$'s in each successive (top-to-bottom) row is nonincreasing, and reorder the columns so that the number of $+$'s in each column (left to right) is nonincreasing. Now, the result will be in the form (3) with $B_{12} < 0$ and $B_{21} > 0$, if it can be so put (and values of m_1 and n_1 can easily be ascertained), i.e. if the original matrix was strictly complementary. This, in addition, yields a pair of permutations if there is one. (This procedure resulted from discussions with F. T. Leighton.) Unfortunately, however, it does not appear that there is a correspondingly simple (i.e. counting based) analog for recognition of (nonstrict) complementarity.

NOTE 4. Suppose $B = (b_{ij})$ is a given n-by-n sign-pattern matrix. For an index set $\gamma \subseteq N \equiv \{1, 2, \ldots, n\}$, define

$$P(\gamma) \equiv \{ j \in N : b_{ij} = + \text{ for some } i \in \gamma \}$$

and

$$M(\gamma) \equiv \{ j \in N : b_{ij} = - \text{ for some } i \in \gamma \}.$$

Recall that B is complementary if and only if there is a proper pair of index sets α, β such that

$$B(\alpha, \beta') \leqslant 0 \quad \text{and} \quad B(\alpha', \beta) \geqslant 0.$$

Now,

$$B(\alpha, \beta') \leqslant 0 \quad \text{iff} \quad \beta' \subseteq P(\alpha)'$$

and

$$B(\alpha', \beta) \geqslant 0 \quad \text{iff} \quad \beta \subseteq M(\alpha')'$$

With these observations, another equivalent condition for a pattern to be complementary is easily demonstrated.

THEOREM 3. *The n-by-n sign pattern B is complementary if and only if either*

(i) *B has a column with no + or a column with no −, or*
(ii) *there is an index set α, $\varnothing \subsetneqq \alpha \subsetneqq N$, such that*

$$P(\alpha) \cap M(\alpha') = \varnothing.$$

In fact, if (ii) holds, then α and $\beta = P(\alpha)$ form a proper pair of index sets which exhibits the fact that the pattern B is complementary. Item (ii) may be restated in the following especially simple form: there is no column of B with both a + entry in a row from α and a − entry in a row from α'. Although item (i) is easily checked, it is not immediately clear how to produce an index set α, as in (ii), if there is one. These ideas can be recast in terms of the usual directed graph on n nodes with a + or − appended to each edge which appears, but this seems to contribute no further insight.

Theorem 3 and Theorem 1 yield another characterization of fully indecomposable inverse nonnegative sign patterns as

COROLLARY 2. *Suppose that B is an n-by-n fully indecomposable sign-pattern matrix. Then, B is inverse nonnegative if and only if* (a) *each column includes a + and a −, and* (b) *for each partition of the rows into two nonempty subsets α and α', there is a column with a + in an α-row and a − in an α'-row.*

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